Draft of Equilibrium
In pursuit of efficient production

By

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PREFACE

The thesis *Draft of Equilibrium* was created at the same time as *Of Absolute Value and the Value in Exchange of Commodities*[1]. In both theses, one concept plays an important role: the result, the income a firm earns from the sale of commodities minus all costs of intermediary supplies, expressed per worker employed by the firm. It enables the measurement of technical change as shown in the paper *A reappraisal of Ricardo's Principles - On measuring technical change*. In this paper we show how firms may use their result per worker in pursuit of efficient production.

Your writer was very much impressed by Gerard Debreu's *Theory of Value*[2]. This paper follows his treatment not only of commodities and prices but also of consumers. However, the big difference concerns producers. The behaviour of producers will be described more explicitly. First of all, by showing how they can produce a given consumer demand as efficiently as possible. Subsequently, it becomes clear how, in competition with each other and in interaction with the prices established on the market, they agree upon equilibrium prices that enable not only to meet the consumer demand, but also to produce it as efficiently as possible. They reach these equilibrium prices by taking over labour from each other as long as their result per worker has not become the same.

The description of the production technologies open to the producers is in line with Sraffa's description of production of commodities by means of commodities[3]. It emphasizes that there is only one indispensable factor of production, namely, labour.

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[1] Both theses, with the Dutch titles 'Schets van Evenwicht' and 'Van Absolute Waarde en de Ruilwaarde van Goederen', were completed in March 1974.

[2] G. Debreu, *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*, 1959, Monograph 17, Cowles Foundation, John Wiley & Sons, New York

[3] P. Sraffa, *Production of Commodities by Means of Commodities*, Cambridge 1960.

You may have noticed by now that the analytical framework of this paper looks similar to the framework underlying the measurement of technical change. But there also appears to be a major difference in the further elaboration of these two theses. In case of measuring technical change much emphasis has been placed on the evidence that this measurement may reveal, namely the evidence that production in accordance with the neoclassical theory does not necessarily imply efficient production. It refers to another paper that shows how oligopolistic competition may easily arise. However, the present paper opens the possibility that oligopolistic competition is only a transitional phase in economic development.

A new phase begins as soon as all producers compete with each other in order to maximize their result per worker so that they can therefore achieve the most efficient production. A proposition that is missing in the original thesis, but which does seem very plausible, is that the resulting equilibrium prices also reflect the embodied direct and indirect labour. It is therefore important to reiterate David Ricardo's first principle:

> *The value of a commodity, or the quantity of any other commodity for which it will exchange, depends on the relative quantity of labour which is necessary for its production, and not on the greater or less compensation which is paid for that labour.* [4]

While the paper *On measuring technical change* shows how to determine where production conditions change and which commodities require more or less direct and indirect labour in their production over time, this paper shows that the prices of the commodities marketed by firms that strive for the most efficient production, according to Ricardo's first principle, will also reflect the relative quantity of labour which is necessary for their production. And equally important, as predicted by Ricardo, the findings of both papers are independent of (changes in) the income distribution.

[4] D. Ricardo, *On the Principles of Political Economy and Taxation*, Volume I of 'The Works and Correspondence of David Ricardo', ed. by P. Sraffa, Cambridge, 1951, p. 11.

In pursuit of efficient production

1. INTRODUCTION

Two reasons are particularly significant to underline the importance of general equilibrium analysis.

The first reason concerns the description of all possible production processes and how they develop over time. Within the production possibilities for each individual commodity, it is important to distinguish how production can take place as efficiently as possible. But because the production of an individual commodity may also require the input of other commodities, efficient production can only be determined as part of the whole of production of an economy. In this paper we follow Nikaido with a definition of efficiency that 'is based solely on the comparison of physical quantities of inputs and outputs'[5]. We can therefore only determine for the economy as a whole whether one net production is produced more efficiently than the other. Unless a production sector of an individual commodity is looked at in more detail and more than one firm is distinguished as a producer. In that case, it can be determined which firm produces relatively most efficiently.

The second reason concerns the description of the competition between producers in interaction with the prices established in the market. This paper suggests that producers should expand or contract their production as long as the result per worker is greater or less than the average result. The result per worker is equal to the yield of the producer's output minus the cost of all necessary inputs, divided by the number of its workers. It is shown that under such competition, consumer demand will eventually be produced efficiently.

The question is whether such an outcome may also be achieved under different competition mechanisms. For example, a mechanism that is mainly aimed at producers who pay their workers a wage so that they can make the greatest possible profit by using their labour in their production processes.

In pursuit of efficient production, the next section describes the reasoning as laid down in the thesis *Draft of equilibrium*. For a simple economy it is proven how efficient production is possible and how this production is within reach through a competitive mechanism in which firms maximize their result per worker. This paper follows the thesis of the time, but has a counterpart in the recent paper *Marshallian Production evokes Schumpeterian Production* that provides a more detailed description of the economy, taking into account, among other things, capital commodities that play a role in production processes over several periods of time. A starting point here is the neoclassical theory in which firms keep their capital commodities in use as long as the marginal revenues exceed the marginal costs. Then it is investigated whether one or more firms, which initially produce in the same way as their direct competitors, under prevailing wages and prices, can make more profit by starting to produce relatively more efficiently. What conditions make this possible?

[5] H. Nikaido, *Convex Structures and Economic Theory*, 1968, Academic Press, New York, p. 185.

2. DRAFT OF EQUILIBRIUM

The economy $\mathcal{E}$ consists of firms that produce a finite number of commodities *i* (*I = 1,…,n*) in successive periods of time *t*. The production takes place in production processes $(\lambda_i^t, x_i^t, y_i^t)$ that describe for each firm producing commodity *i* its output in period *t* in relation to the input of commodities, together with the input of labour supplied by households, where

λ_i^t is a nonnegative scalar, representing the use of homogenous labour,

x_i^t an *n*-dimensional nonnegative input vector of intermediate consumption, and

y_i^t an *n*-dimensional nonnegative output vector.

All technologically possible production processes for commodity *i* build up the production technology set T_i^t. The assumptions on T_i^t are as follows:

ASSUMPTION 1: *$(\lambda_i^t, x_i^t, y_i^t) \in T_i^t$ and $\lambda_i^t = 0$ imply $y_i^t = 0$. Labour is indispensable.*

ASSUMPTION 2: *T_i^t is closed and contiguous for each of its positive intersections, i.e. if $(\lambda_i^t, x_i^t, y_i^t) \in T_i^t$ then there are also processes (μ_i^t, u_i^t, v_i^t) with values positive for u_i^t and v_i^t where they are also positive in x_i^t and y_i^t for every μ_i^t from the interval $(0, \lambda_i^t)$.*

ASSUMPTION 3: *$y_{i,j}^t = 0$ for every $j \neq i$. No joint production.*

ASSUMPTION 4: *$(\lambda_i^t, x_i^t, y_i^t) \in T_i^t$ and $u_i^t \geq x_i^t$, $y_i^t \geq v_i^t \geq 0$ implies $(\lambda_i^t, u_i^t, v_i^t) \in T_i^t$. Free disposability.*

These assumptions reflect reality, as is evident with the indispensability of labour, or are made for reasons of simplicity. The closeness of T_i^t also reflect reality as it implies that if a sequence of possible production processes approaches to $(\lambda_i^t, x_i^t, y_i^t)$ then $(\lambda_i^t, x_i^t, y_i^t)$ also belongs to T_i. Somewhat more complex is the second part of assumption 2. It says, for example, that if the use of more labour yields increasing returns, the way back is easier. If twice as much labour initially yields three times more production, half the production can also be achieved if labour is halved later on. So, once a production level is reached, a convex cone applies to the lower production levels.

Assumption 3, like 4, is merely a simplification. After all, joint production can be regarded as a commodity that consists of two separate parts. If these separate parts can also be produced separately, the search for efficiency that takes shape by equalizing the result per worker will show which one is most efficient. If only joint production remains, the price to be determined for the separate parts will then depend on the demand for those commodities, although the result per worker will still determine the price of both commodities together.

The implicitly assumed homogeneous labour is also not essential. In the case of heterogeneous labour, the search for efficiency will initially focus mainly on the result per worker between firms that produce the same commodity. For the economy as a whole, it is then necessary to consider how the supply of the various types of labour reacts to the differences in the result per worker in the various sectors of the economy. In this case, differentiated wages also become relevant as determinants of labour supply.

Regardless of the assumptions, it should be emphasized that different firms may produce the same commodity while using different production processes. They may, for example, differ in the input of intermediate consumption. But here too, one method may prove to be more efficient than the other.

The union of all sets T_i^t defines the overall technology set of the economy at period *t*.

$$T^t = \bigcup_i T_i^t,$$

which consists of production processes

$$(\lambda^t, x^t, y^t) = \sum_i(\lambda_i^t, x_i^t, y_i^t).$$

Within set T^t with an almost infinite number of production processes that are allowed by the total labour supply ω^t we define the set T_+^t that consists of net output vectors $(y\text{-}x)^t$ where the production vectors y^t are greater than or equal to the required input vectors x^t.

$$T_+^t = \{(\lambda^t, (y-x)^t) \mid (y-x)^t \geq 0\,, (\lambda^t, x^t, y^t)\ \in T^t\}$$

The economy $\mathcal{E} = [(T_1^t, \ldots, T_n^t), \omega^t]$ is called productive when T_+^t is not empty,

Of course we would like to know the maximum possible output that the labour supply allows to produce, where the maximum possible output consists of an almost infinite number of net output vectors. To streamline the search for efficient production, we use the following definitions:

DEFINITION 1
A production process (λ^t, x^t, y^t) is called more efficient than another process (μ^t, u^t, v^t) if

$$\begin{pmatrix} -\lambda^t \\ (y-x)^t \end{pmatrix} \geq \begin{pmatrix} -\mu^t \\ (v-u)^t \end{pmatrix}.$$

DEFINITION 2
A production process $(\lambda^t, x^t, y^t) \in T_+^t$ is called efficient if there is no more efficient process in T_+^t than (λ^t, x^t, y^t).
DEFINITION 3
A subset of T_+^t is said to be efficient with respect to T_+^t if no production process in T_+^t minus the subset is more efficient than a process from that subset.

The first two definitions are taken from Nikaido. This also applies to the first part of the proof of the efficiency theorem below, which is corresponds to Nikaido's proof of the nonsubstitution theorem[6]. This concerns the proof that the set $T_+^t(\omega^t)$ described below is compact[7].

THEOREM 1 (efficiency)

If the economy $E = [(T_1^t, \ldots, T_n^t), \omega^t]$ is productive, then there exists a non-empty set $\dot{T}_+^t(\omega^t)$ that is efficient with respect to T_+^t, i.e. a set of net output vectors $(y\text{-}x)^t$, the production of which takes up the whole supply of labour ω^t while no $(y\text{-}x)^t$ with less labour can be produced. Moreover, each production sector i is capable of producing exactly the quantities of commodities stated in the net output vectors $(y_i\text{-}x_i)^t$.

Proof:
The set of net output vectors allowed by labour supply ω^t is described by

$$T_+^t(\omega^t) = \{(\lambda^t, (y-x)^t) \mid \lambda^t \leq \omega^t, (y-x)^t \geq 0\,, (\lambda^t, x^t, y^t)\ \in T^t\}.$$

The set $T_+^t(\omega^t)$ is not empty because the economy is productive.

Before defining $\dot{T}_+^t(\omega^t)$, we first prove that $T_+^t(\omega^t)$ is compact. To this end, we look at the set of net output vectors that are greater than or equal to any vector c^t, $c^t \in T_+^t(\omega^t)$, so that holds

$$T_+^t(\omega^t, c^t) = \{(\lambda^t, (y-x)^t) \mid \lambda^t \leq \omega^t,\ (y-x)^t \geq c^t\,, (\lambda^t, x^t, y^t)\ \in T^t\}.$$

Because c^t is part of it, $T_+^t(\omega^t,c^t)$ is not empty. It is also clear that $T_+^t(\omega^t,c^t) \subset T_+^t(\omega^t) \subset T_+^t$. To prove the compactness of $T_+^t(\omega^t)$ it suffices that $T_+^t(\omega^t,c^t)$ is closed and bounded.

[6] *Op. cit.*, see p. 185 and p. 190.
[7] Compactness results from the indispensable labour. As stated by Georgescu-Roegen, compactness has to be assumed if "labour, while an indispensable factor in production of any commodity, can be substituted by other factors beyond any limit". Such a substitution clearly conflicts with reality. See N. Georgescu-Roegen, *Analytical economics*, 1966, Cambridge: Harvard University Press.

If $T_+^t(\omega^t,c^t)$ were unbounded, there would be a sequence $\{(\lambda_v^t, x_v^t, y_v^t)\}$ in $T_+^t(\omega^t,c^t)$ such that lim $\| y^v \|$ = $+\infty$. By decomposing each term of the sequence to its individual sectors

$$\lambda_v^t = \sum_{i=1}^n \lambda_{iv}^t, \qquad y_v^t = \sum_{j=1}^n y_{iv}^t, \qquad x_v^t = \sum_{j=1}^n x_{iv}^t, \tag{1}$$

we obtain sequences $\{(\lambda_{iv}^t, x_{iv}^t, y_{iv}^t)\}$ in T_i^t. From $y_v^t \geq x_v^t$, $\lambda_{iv}^t \geq 0$, $x_{iv}^t \geq 0$, $y_{iv}^t \geq 0$ and equation (1) follow

$$0 \leq \lambda_{iv}^t \leq L^t \quad (i = 1, \dots, n),$$

$$\|x_{iv}^t\|, \|y_{iv}^t\| \leq \|y_v^t\| = \eta_v \quad (i = 1, \dots, n).$$

Whence $\{(\lambda_{iv}^t/\eta_v, x_{iv}^t/\eta_v, y_{iv}^t/\eta_v)\}$ is a bounded sequence in T_i^t for each i, because T_i^t is contiguous for each of its positive intersections. Hence, without loss of generality, these n bounded sequences may be assumed to converge to (μ_i^t, u_i^t, v_i^t), respectively. Note that $(\mu_i^t, u_i^t, v_i^t) \in T_i^t$ by the closedness of T_i^t But, $0 \leq \lambda_{iv}^t \leq \omega^t$, lim $\eta_v = +\infty$ imply $\mu_i^t = 0$ $(i=1,\dots,l)$, so that $v_i^t = 0$ $(i=1,\dots,l)$ by the indispensability of labour in T_i^t. Therefore we must have

$$\lim_{v \to +\infty} \frac{1}{\eta_v} \sum_i^n (\lambda_{iv}^t, x_{iv}^t, y_{iv}^t) = \sum_i^n (\mu_i^t, u_i^t, v_i^t) = \left(0, \sum_i^n u_i^t, 0\right),$$

which implies

$$\lim_{v \to +\infty} \frac{1}{\eta_v} \sum_i^n (y_{iv}^t) = 0.$$

This contradicts

$$\left\| \frac{1}{\eta_v} \sum_i^n y_{iv}^t \right\| = 1 \quad (v = 1,2, \dots),$$

proving the boundedness.

Next let us prove the closedness of $T_+^t(\omega^t,c^t)$. If $\{(\lambda_v^t, y_v^t\text{-}x_v^t)\}$ is a sequence in $T_+^t(\omega^t,c^t)$ converging to (λ^t,d^t), the sequences $\{(\lambda_{iv}^t, x_{iv}^t, y_{iv}^t)\}$ in T_i^t obtained by the corresponding decomposition (1) are bounded from the boundedness of $\{y_v^t\}$shown above in conjunction with $\omega^t \geq \lambda_v^t \geq 0$, $y_v^t \geq y_{iv}^t \geq 0$, $y_v^t \geq x_v^t \geq x_{iv}^t \geq 0$. Whence we may assume, without loss of generality, that they are originally convergent, with their respective limits $(\lambda_i^t, x_i^t, y_i^t)$. Then we have

$$(\lambda^t, d^t) = \lim_{v \to +\infty} (\lambda_v^t, y_v^t - x_v^t) = \lim_{v \to +\infty} \sum_{i=1}^n (\lambda_{iv}^t, y_{iv}^t - x_{iv}^t) = \sum_{i=1}^n (\lambda_i^t, y_i^t - x_i^t).$$

We now observe that $(\lambda_i^t, x_i^t, y_i^t) \in T_i^t$ by the closedness of T_i^t. Moreover, $\lambda_v^t \leq \omega^t$, $y_v^t\text{-}x_v^t \geq c^t$ $(v=1, 2, \dots)$ ensure $\Sigma\, \lambda_i^t \leq \omega^t$, $\Sigma\, (y_i^t\text{-}x_i^t) \geq c$. Whence $(\lambda^t,d^t)=\Sigma\, (\lambda_i^t, y_i^t\text{-}x_i^t) \in T_+^t(\omega^t,c^t)$, proving the closedness of $T_+^t(\omega^t,c^t)$.

Next we construct a function $f : T_+^t(\omega^t,c^t) \to R$, where $f(\lambda^t, y^t\text{-}x^t)= \lambda^t$ for all $(\lambda^t, y^t\text{-}x^t) \in T_+^t(\omega^t,c^t)$. Then, by its continuity, function f has a minimum according to the theorem of Weierstrass. The complete original of this minimum λ^t is the set $\dot{T}_+^t(\omega^t,c^t)$. Thus, for every c^t, $c^t \in \dot{T}_+^t(\omega^t,c^t)$, we get a set in which c^t is produced with a minimum of labour.

Again we construct another function

$$g : \bigcup_{c^t, c^t \in T_+^t(\omega^t)} \dot{T}_+^t(\omega^t, c^t) \to R,$$

where $g(\lambda^t, y^t\text{-}x^t)= \lambda^t$ for all $(\lambda^t, y^t\text{-}x^t) \in \dot{T}_+^t(\omega^t,c^t)$. Again, it is clear that g is continuous so that the function reaches a maximum. The complete original of this maximum λ^t is the intended set $\dot{T}_+^t(\omega^t)$. $\dot{T}_+^t(\omega^t)$ is clearly efficient with respect to $T_+^t(\omega^t)$, but it is also efficient with respect to T_+^t. Namely, suppose

$$\begin{pmatrix} -\lambda^t \\ (y-x)^t \end{pmatrix} \geq \begin{pmatrix} -\mu^t \\ (v-u)^t \end{pmatrix},$$

with $(\lambda^t, y^t\text{-}x^t) \in T_+^t$ and $(\mu^t, u^t, v^t) \in \dot{T}_+^t(\omega^t)$, which means $\lambda^t \leq \mu^t = \omega^t$, from which follows $(\lambda^t, y^t\text{-}x^t) \in T_+^t(\omega^t)$.

The last statement of the theorem follows directly from the assumption of free disposability.
Q.E.D.

We will now expand the economy with n-dimensional non-negative price vectors p^t that attach a price to each commodity i. Within a productive economy, for each price system p^t for each sector i, the result per worker can be determined in period t. This is represented by res_{ip}^t

$$res_{ip}^t = \frac{p^t\,(-x_{i,1}^t,\dots,-x_{i,i-1}^t, y_i^t, -x_{i,i+1}^t,\dots,-x_{i,n}^t)}{\lambda_i^t}.$$

For the total result of all sectors together res_p^t applies: $res_p^t = \Sigma_i \lambda_i^t res_{ip}^t$.

A productive economy ensures that λ_i^t is greater than zero. After all, then it holds that $y^t \geq x^t$ such that either $y_i^t > x_i^t$, so that $\lambda_i^t > 0$, or $y_i^t = x_i^t$ where y_i^t serves as an intermediate product, so that here too $\lambda_i^t > 0$. If $y_i^t = 0$, this commodity is not part of the productive economy.

We also impose the restriction on the price systems that the value of the net output vector c^t under each permitted price system is equal to an amount K that remains the same in time:

$$p^t c^t = K.$$

For the productive economy $\mathcal{E} = [(T_1^t, \dots, T_n^t), \omega^t, K]$ the set of allowed price systems P_s^t can be represented by

$$P_s^t = \{p^t \in R^n | (\omega^t, c^t) \in T_+^t,\ p^t c^t = K \ and\ \forall i:\ res_{ip}^t \geq 0\}.$$

Conversely, we then define the set Y_s^t

$$Y_s^t = \{(\omega^t, c^t) \in T_+^t | \forall p^t \in P_s^t\}.$$

Competition between firms appears from the following two definitions:

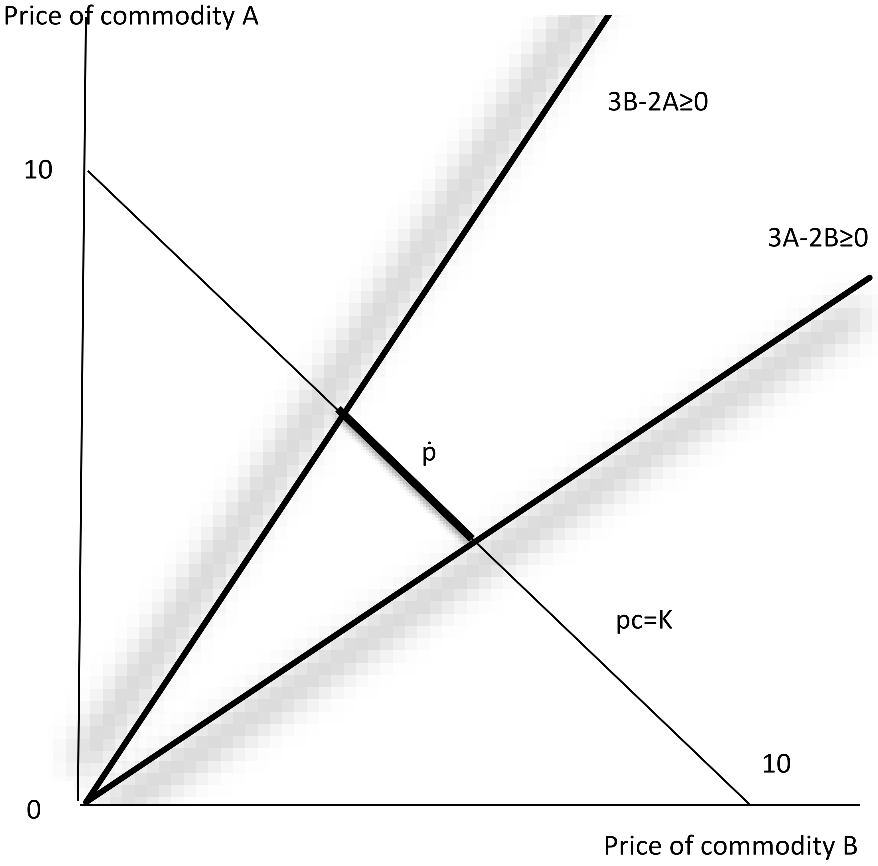
DEFINITION 4
A firm can take over labour from another firm if its result per worker is greater than in the other firm.
DEFINITION 5
The aggregate of firms is said to be stable relative to a price system if no firm can take over labour from another firm.

To prove that there is a price system under which no firm can take over labour from another firm, it must first be proved that the set P_s^t is not empty. To clarify the proof in the lemma below, we first give an example of an economy that produces two goods A and B, with a net output $c = (1,1)$ and $K = 10$. The following techniques are used to produce this net output.

A) 3A+2B→6A	}	3A-2B≥0
B) 2A+2B→5B	}	3B-2A≥0

The techniques show that to produce net output 3A an input of 2B is required, while a net production of 3B requires an input of 2A. In figure 1 two half spaces are distinguished within which a positive result is achieved for each p. The intersection of these half spaces with the price plane $pc=K$ then determines the price systems $\dot{p}$ that yield a positive result for both sectors.

Figure 1



LEMMA 1

If the economy $\mathcal{E} = [(T_1^t, \ldots, T_l^t), \omega^t, K]$ is productive, then there exists for each $c^t \in T_+^t(\omega^t)$ a non-empty set of price vectors $\dot{p}^t$ for which for every price vector $p^t \in \dot{p}^t$ holds $p^t c^t = K$.

Proof:
The n production techniques used to produce one $c^t \in T_+^t(\omega^t)$ involve half spaces for which holds

$$\forall i: \; -x_{i,1}^t - x_{i,2}^t - \cdots - x_{i,i-1}^t + y_i^t - x_{i,i+1}^t - \cdots - x_{i,n}^t \geq 0$$

with

$$c^t = \sum_{i=1}^{n} (y_i^t - x_i^t) = \sum_{i=1}^{n} \left(y_i^t - \sum_{j=1, j \neq i}^{n} x_{i,j}^t \right)$$

The intersection of these half spaces is a cone through the origin. This cone is not empty, otherwise production y^t would be less than intermediate consumption x^t. Therefore, the intersection with hyperplane $p^t c^t = K$ is not empty either.
Q.E.D.

Next, we construct the set of price vectors C^t

$$\mathrm{C^t} = \bigcup_{\mathrm{c, c \in T_+^t(\omega^t)}} \dot{\mathrm{p}}^{\mathrm{t}}$$

We will now prove that the sets C^t and P_s^t are equal.

LEMMA 2

If the economy $\mathcal{E} = [(T_1^t, \ldots, T_l^t), \omega^t, K]$ is productive, then $C^t \supset P_s^t$ and $P_s^t \supset C^t$.

Proof:
Each p^t from C^t corresponds to a net output vector $c^t \in T_+^t(\omega^t)$, for which $p^t c^t = K$ and

$$\forall i : p^t \, (-x_{i,1}^t, \ldots, -x_{i,i-1}^t, y_i^t, -x_{i,i+1}^t, \ldots, -x_{i,n}^t) \geq 0.$$

Since the economy is productive, $\lambda_i^t > 0$ so that

$$\forall i : \frac{p^t \, (-x_{i,1}^t, \ldots, -x_{i,i-1}^t, y_i^t, -x_{i,i+1}^t, \ldots, -x_{i,n}^t)}{\lambda_i^t} \geq 0.$$

Thus p^t satisfies the conditions of P_s^t.

Conversely, every p^t from P_s^t satisfies the conditions for admission to C^t.
Q.E.D.

We are going to expand the economy again. We assume that the households also have an amount to spend to the *n* commodities that is equal to *K* every period. In line with Debreu, we further assume that there exits *n*-dimensional non-negative price vectors p_d^t that generate *n*-dimensional non-negative demand vectors d^t where $p_d^t d^t = K$ also holds. All possible price vectors form the set P_d^t, while all possible demand vectors are part of the set Y_d^t.

Next, we define what is meant by equilibrium.

DEFINITION 6
An equilibrium is a price system relative to which the aggregate of firms is stable and under which they produce a net output vector as efficiently as possible that meets the final demand associated with that price system.

It is clear that an equilibrium must be found in the intersection of the set P_s^t and P_d^t, and that the net output vector that satisfies the final demand belongs to the intersection of Y_s^t and Y_d^t. That implies the sets

$$P_*^t = P_s^t \cap P_d^t \qquad \text{and} \qquad Y_*^t = Y_s^t \cap Y_d^t.$$

The following theorem, known as the theorem of Kuratowski, Knaster and Mazurkiewicz (see Berge[8] and Kuratowski [9]), accounts for the core of the proof that an equilibrium exists indeed.

If $A_0, \ldots, A_n$ are (n+1) closed sets in simplex S such that each face $p_{i0} \ldots p_{ik}$ $(0 \leq k \leq n)$ of the simplex satisfies

$p_{i0} \ldots p_{ik} \subset A_{i0} \cup \ldots \cup A_{ik}$

then $A_0 \cap \ldots \cap A_n = \emptyset$.

A corollary of this theorem got an earlier application in the preliminary proof of fair net trades by Schmeidler and Vind[10], although they changed their proof later on[11].

[8] Berge,
[9] K. Kuratowski, Topology I, 1966, Academic Press, New York..
[10] D. Schmeidler and K. Vind, *Fair net trades*, 1970, Discussion Paper 7014, CORE Louvain.
[11] D. Schmeidler and K. Vind, *Fair net trades*, 1972, Econometrica, 40, 637-642.

THEOREM 2 (existence)

If economy $E = [(T_1^t, \dots, T_l^t), \omega^t, K]$ is productive, then there exists an equilibrium.

Proof:
Because the economy is productive and $p^t c^t = K$, the sets Y_*^t and P_*^t are not empty. We are going to prove the existence of an equilibrium in the following subset of P_*^t, in which the result per worker for all sectors i and for all $c^t \in Y_*^t$ reaches a maximum:

$$Q^t = \{p^t \in P_*^t \,|\, (c^t) \in Y_*^t \text{ and } \forall \dot{p} \in P_*^t : [\forall i : res_{i\dot{p}}^t \geq res_{ip}^t] \text{ implies } [\forall i : res_{i\dot{p}}^t = res_{ip}^t]\}$$

Since P_*^t and Y_*^t are not empty, Q^t is not empty either. We then define for each sector j $(j=1,\dots,n)$ the price systems in Q^t under which the result per worker in sector j is the lowest of all sectors.

$$M_j^t = \{p^t \in Q^t \,|\, (c^t) \in Y_*^t \text{ and } \forall i : res_{ip}^t \geq res_{jp}^t\}$$

The economy is stable with respect to a price system from Q^t if it belongs to $\cap_{j,j=1,\dots,n} M_j^t$. To prove that this intersection is not empty, we are going to embed Q^t in the simplex S in R^n.

$$S = \{s \in R^n | \forall i : s_i \geq 0 \text{ and } \textstyle\sum_i s_i = 1\}$$

For every i we denote by S_i the i^{th} face of S, i.e.

$$S_i = \{s \in S | s_i = 0\}$$

We need the following application of Sperner's lemma:

LEMMA 3 (Kuratowski, Knaster and Mazurkiewicz)

For each i let Q_i^t be a closed set in S containing S_i. If $\bigcup_i Q_i^t = S$ then $\bigcap_i Q_i^t \neq \emptyset$.

In order to embed Q^t in S we define the mapping of $p^t \in Q^t$ for each corresponding $c^t \in Y_*^t$ to $\prod(p^t)$ in S by:

$$\prod(p^t) = \frac{res_{1p}^t, \dots, res_{ip}^t, \dots, res_{np}^t}{\sum_i res_{ip}^t}$$

For each $p^t \in Q^t$ the denominator is greater than zero because the economy is productive. Moreover, $p^t c^t = K$ represents the intersection of a hyperplane with a convex cone which implies the continuity of p^t given c^t. Therefore, $\prod$ is continuous. The sets $\{M_i^t\}$ are clearly closed. Hence, the sets $\{\prod(M_i^t)\}$ are also closed in S. The definition of p^t and the assumptions on $\{M_i^t\}$ imply that for each i holds $\prod(M_i^t) \supset S_i$. After all, all elements res_{kp}^t, $k \neq i$, are greater than or equal to res_{ip}^t and res_{ip}^t can be equal to zero.

To complete the proof of theorem 2 we have to proof that

(i) $\bigcup_i M_i^t = Q^t$

Assume, per absurdum, that there is a p^t in Q^t minus $\bigcup_i M_i^t$. Choose i_1. As p^t is not in M_{i1}^t, there is i_2 for which $res_{i1p}^t > res_{i2p}^t$. But $p^t \notin M_{i2}^t$, so there is i_3 with $res_{i2p}^t > res_{i3p}^t$. Proceeding by induction we construct a sequence $i_1, i_2, \dots, i_k, \dots$ so that the result per worker of sector k is smaller than that of sector $k\text{-}1$. After at most $n+1$ steps, a cycle $i_1, i_2, \dots, i_n = i_1$ results where $i_k \neq i_j$ for $j,k=1, \dots, n\text{-}1$ and $k \neq j$, which is a contradiction.

(ii) $\prod(Q^t) = S$

Because the closed set Q^t is not empty, $\Pi(Q^t)$ is also a closed set in S. Because for each $p^t \in Q^t$ and for each i holds that $res_{ip}{}^t \geq 0$ is also $\sum_i res_{ip}{}^t$ greater than or equal to zero. Therefore,

$$\forall i: \quad s_i = \frac{res_{ip}^t}{\sum_i res_{ip}^t} \geq 0 \quad \text{while} \quad \sum_i s_i = \sum_i \frac{res_{ip}^t}{\sum_i res_{ip}^t} = 1.$$

Q.E.D.

The price established on the market for each commodity i can be represented by

$$p_i^t = \frac{\lambda_i^t res_{ip}^t - \sum_{j,j\neq i} \frac{\lambda_j^t res_{jp}^t}{q_{i,j}^t}}{q_i^t},$$

where q_i denotes the quantity of commodity i produced and q_j the required input of commodity j.

THEOREM 3 (equilibrium prices)

A price system of an equilibrium reflects the input of direct and indirect labour.

Proof:

In an equilibrium, the result per worker is by definition the same in all firms. The price of each commodity i which is equal to

$$p_i^t = \frac{\lambda_i^t res_{ip}^t - \sum_{j,j\neq i} \frac{\lambda_j^t res_{jp}^t}{q_{i,j}^t}}{q_i^t},$$

can then be rewritten as

$$\frac{p_i^t}{res_{ip}^t} = \frac{\lambda_i^t - \sum_{j,j\neq i} \frac{\lambda_j^t}{q_{i,j}^t}}{q_i^t}.$$

These equilibrium prices obviously reflect the embodied direct and indirect labour.

Q.E.D.

We will see elsewhere[12] that this theorem also applies to a more complex economy that includes the use of capital commodities.

3. FUTURE WORK

The above results are obtained under the assumption of homogeneous labour. But in the case of heterogeneous labour that is paid differently, comparable results seem possible. The definition of efficiency remains relevant

A production process *(λ^t, x^t, y^t)* is called more efficient than another process *(μ^t, u^t, v^t)* if

$$\begin{pmatrix} -\lambda^t \\ (y-x)^t \end{pmatrix} \geq \begin{pmatrix} -\mu^t \\ (v-u)^t \end{pmatrix}.$$

[12] A. Moons, *Marshallian Production evokes Schumpeterian Production*, 2023.

This definition applies to all production processes together, even if the total available labour consists of a number of categories that are paid differently. However, It is important that these different categories are used as efficiently as possible. In case of differences in production technology, this can lead to more use being made of a certain (relatively scarce) category of labour in one sector and less in the other, viewed from the perspective of the total economy. For the question of whether a firm can take over labour from another firm, the simple result per worker will then no longer suffice, but must be replaced by a relative result. This relative result per worker is the simple result per worker compared to the result per worker that a firm has to pay for the use of the different categories of labour. In addition to the market prices of commodities, use must therefore also be made of the wage of the various categories of labour as they are established on the market. Definition 4 then becomes

A firm can take over labour from another firm if its result per worker,
in comparison with the average wages required, is greater than in the other firm.

With these adjustments, it seems that the above theorems can be proven again for an economy with heterogeneous labour.

In addition, the competition mechanism described above raises the question of whether there are already firms that actually strive for a maximum relative result per worker. It is conceivable that, for example, in the service sector there are firms that would like to know whether there are competitors who achieve a higher result per worker than themselves. But in the manufacturing industry that often works with capital goods supplied by other firms, the costs of using labour in addition to capital will be the main focus.

Referring to the paper *Marshallian Production evokes Schumpeterian Production*, the Introduction asks what circumstances make it possible for Schumpeterian firms to emerge in the midst of Marshallian firms. The most important condition is a profit rate that is higher than necessary to finance the investments required to maintain the production capacity. This condition implies technological change over time. In an economy without technical innovation, the Golden Rule of Accumulation would apply, under which production is by definition efficient. The mentioned paper will show that just technical change is a precondition for Schumpeterian firms to emerge.

As the market share of Marshallian firms decreases, it becomes more important to compare the Schumpeterian firms with each other. Where production is most efficient will become apparent from the (relative) result per worker. In comparable market conditions, both reach the highest values at the same time.